On the Numerical Errors Induced by the Space-Time Discretization in the LE-FDTD Method

F. Alimenti and L. Roselli, *Senior Member, IEEE*

***Abstract*—In this letter, the accuracy of the lumped element finite-difference time-domain (LE-FDTD) method is discussed in the particular case of a planar distribution of equal resistors. Following the von Neumann technique and assuming a uniform grid, the effective impedance of the lumped resistor has been rigorously derived in a closed form. The result obtained has been compared with the LE-FDTD simulation of a simple test structure. This structure consists of an infinitely long parallel-plate waveguide loaded with the planar distribution of resistors. The excellent agreement obtained validates the approach showing a dependence of the effective resistor impedance on spatial and temporal discretization steps.**

***Index Terms*—Finite-difference time-domain (FDTD), lumped elements, numerical parasitics.**

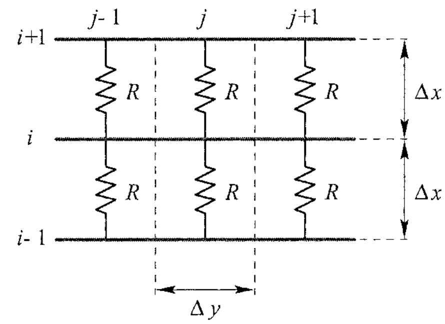

Fig. 1. Cross-sectional view of the resistive layer: each resistor R is placed on a different FDTD cell.

I. INTRODUCTION

LUMPED elements can be incorporated into the finite-difference time-domain (FDTD) framework to effectively describe electronic devices and networks, the dimensions of which are smaller than the wavelength of the exciting signal [1], [2].

The response of such elements, however, deviates from ideality because of numerical errors [3]. First, in the FDTD analysis of a high-frequency electronic circuit, the unavoidable numerical discontinuities associated to the junction between lumped element and distributed structures must be accounted for. For example, if an interconnection line is terminated with a lumped resistor, a step-like discontinuity is formed at the connecting point. A quantitative estimation of these parasitic effects is very difficult unless suitable de-embedding techniques are adopted [4]. A second cause of numerical errors is the effect of the well-known numerical dispersion of the FDTD lattice on the actual electrical behavior of the lumped element. In this case, the problem concerns the approximation of the Faraday and Ampère equations within the discretized space time domain.

This work aims at analytically estimating the latter kind of error. To this purpose, the particular case of a planar distribution of equal resistors has been considered. As a result, the effective impedance of the lumped resistor is obtained showing a dependence on the spatial and temporal discretization steps.

II. NUMERICAL ERRORS

The scope of this section is to provide an evaluation of the numerical errors associated with the lumped elements, at least in a simple, but meaningful, case. A planar distribution of equal resistors has been considered to form a layer as depicted in Fig. 1. The basic idea is to derive the behavior of the generic resistor embedded in the layer by evaluating the scattering off the layer when a plane wave impinges on it. The same methodology of investigation has been used in the past to derive the numerical errors associated to dielectric [5] or meshing [6] discontinuities.

For the accurate evaluation of the scattering, the resistive layer has been placed across an infinitely long parallel-plate waveguide (PPWG), as illustrated in Fig. 2. In particular, the resistors have been oriented along the x-axis, parallel to the electric field of the fundamental TEM mode propagating in the structure.

The reflection coefficient of the above geometry can simply be evaluated if the resistance R is assumed to be independent from the space time discretization, i.e., if the numerical errors are neglected. The result follows immediately observing that each resistor of the layer can be seen as placed across a single-cell PPWG of width Δy and height Δx. The load of such a PPWG is represented by its characteristic impedance Z_0 in parallel with the resistor R, thus

$$\Gamma_0 = \frac{Y_0 - (Y_0 + 1/R)}{Y_0 + (Y_0 + 1/R)} = -\frac{1}{1 + 2\,\frac{\Delta y}{\Delta x}\,\frac{R}{\eta}} \tag{1}$$

where $Y_0 = 1/Z_0$, $Z_0 = \eta\,\Delta x/\Delta y$ and $\eta = \sqrt{\mu/\epsilon}$ is the wave impedance of the medium.

Nevertheless, experiments carried out by the authors have demonstrated that the reflection coefficient of the structure in Fig. 2 deviates from (1) because of numerical errors. To analyze these effects, the discontinuity problem stated in Fig. 2 must be solved in the discretized space time.

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The authors are with the Dipartimento di Ingegneria Elettronica e dell'Informazione, Università di Perugia, Perugia, Italy (e-mail: alimenti@diei.unipg.it).

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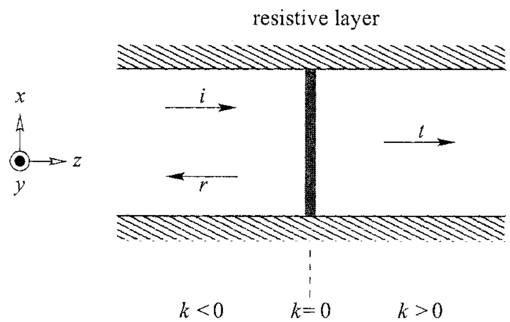

Fig. 2. Infinitely long PPWG loaded with a resistive layer at the discrete z-coordinate $k = 0$.

The lumped element FDTD (LE-FDTD) equation used to update the electric field component E_x on the resistive layer at location $k = 0$ is:

$$\frac{E_x^{n+1}\left(i+\frac{1}{2}, j, 0\right) - E_x^{n}\left(i+\frac{1}{2}, j, 0\right)}{\Delta t} = -\frac{1}{\epsilon}\,\frac{H_y^{n+1/2}\left(i+\frac{1}{2}, j, \frac{1}{2}\right) - H_y^{n+1/2}\left(i+\frac{1}{2}, j, -\frac{1}{2}\right)}{\Delta z} - \frac{\Delta x}{\epsilon R\,\Delta y\,\Delta z}\,\frac{E_x^{n+1}\left(i+\frac{1}{2}, j, 0\right) + E_x^{n}\left(i+\frac{1}{2}, j, 0\right)}{2}. \quad (2)$$

For the rigorous solution of the considered scattering problem, electric and magnetic fields must be written as the superposition of plane waves propagating in the discretized space time domain. Following the von Neumann analysis [7, p. 827], the electric field is assumed to be

$$E_x^n\left(i+\frac{1}{2}, j, k\right) = \begin{cases} E_{x,i}e^{j\Omega n}e^{-j\chi_z k} \\ \quad +E_{x,r}e^{j\Omega n}e^{j\chi_z k}, & \text{if } k \le 0 \\ E_{x,t}e^{j\Omega n}e^{-j\chi_z k}, & \text{if } k > 0 \end{cases} \quad (3)$$

where $E_{x,i}$, $E_{x,r}$, and $E_{x,t}$ are the amplitudes of the incident, reflected and transmitted electric fields. The magnetic field can be expresses in terms of the same quantities as follows:

$$H_y^{n+1/2}\left(i+\frac{1}{2}, j, k+\frac{1}{2}\right) = \begin{cases} \dfrac{E_{x,i}}{\eta}\, e^{j\Omega(n+1/2)}e^{-j\chi_z(k+1/2)} \\ \quad -\dfrac{E_{x,r}}{\eta}\, e^{j\Omega(n+1/2)}e^{j\chi_z(k+1/2)}, & \text{if } k < 0 \\ \dfrac{E_{x,t}}{\eta}\, e^{j\Omega(n+1/2)}e^{-j\chi_z(k+1/2)}, & \text{if } k \ge 0. \end{cases} \quad (4)$$

In the above expressions, Ω and χ_z are the normalized angular frequency and phase constant, respectively

$$\Omega = \omega\,\Delta t; \qquad \chi_z = \beta_n\,\Delta z \quad (5)$$

and β_n is the numerical phase constant of the algorithm. The above quantities satisfy the well-known FDTD dispersion relationship [8]

$$\frac{1}{v\,\Delta t}\,\sin\left(\frac{\Omega}{2}\right) = \frac{1}{\Delta z}\,\sin\left(\frac{\chi_z}{2}\right) \quad (6)$$

$v = 1/\sqrt{\mu\epsilon}$ being the phase velocity of the waves in the considered medium. The amplitudes of incident, reflected and transmitted waves are related by the continuity of the electric field E_x at the layer location $k = 0$

$$E_{x,i} + E_{x,r} = E_{x,t}. \quad (7)$$

In order to compute the reflection coefficient of the resistive layer, the discrete plane wave solutions (3), (4) must be substituted into the LE-FDTD equation (2). Using the continuity of the electric field E_x at the layer location (7), after some manipulations, one obtains

$$-\frac{2v}{\Delta z}\,E_{x,r}e^{-j(\chi_z/2)} + j\,\frac{2\,v}{\Delta z}\,E_{x,i}\sin\left(\frac{\chi_z}{2}\right) = (E_{x,i} + E_{x,r}) \cdot \left[\frac{\Delta x}{\epsilon R\,\Delta y\,\Delta z}\,\cos\left(\frac{\Omega}{2}\right) + j\,\frac{2}{\Delta t}\,\sin\left(\frac{\Omega}{2}\right)\right]. \quad (8)$$

The above formula can now be simplified using the FDTD dispersion relationship (6) and exploited to derive the reflection coefficient

$$\Gamma_R = \frac{E_{x,r}}{E_{x,i}} = -\frac{\cos\left(\frac{\Omega}{2}\right)}{\cos\left(\frac{\Omega}{2}\right) + 2\,\frac{\Delta y}{\Delta x}\,\frac{R}{\eta}\,\cos\left(\frac{\chi_z}{2}\right)}. \quad (9)$$

Once the reflection coefficient has been evaluated rigorously, it can be used to derive the numerical impedance of the layer. This can be done by comparing the expressions of Γ_0 and Γ_R; it emerges that the value of R in (1) is replaced in (9) by

$$R_n = R\,\frac{\cos\left(\frac{\chi_z}{2}\right)}{\cos\left(\frac{\Omega}{2}\right)} \quad (10)$$

which is just the numerical impedance of each lumped resistor embedded in the layer. The obtained result indicates a purely resistive component depending by the space time discretization steps.

III. Validation

In order to validate the above computations, the reflection coefficient of the structure in Fig. 2 has been determined both analytically, using the formula (9), and numerically, adopting the three-dimensional LE-FDTD simulator developed in [9].

In the latter case, the geometry has been discretized with a mesh featuring $\Delta x = 0.44$ mm and $\Delta y = \Delta z = 2$ mm; the time step has been chosen at the Courant's limit and is $\Delta t = 1.4$ ps. Each layer resistance is $R = 83.3\ \Omega$. To obtain a very high accuracy, the PPWG has been excited and terminated with a matched modal source and a modal absorbing boundary condition respectively [10]. As post-processing option, the reference plane has been defined exactly at the location of the resistive layer.

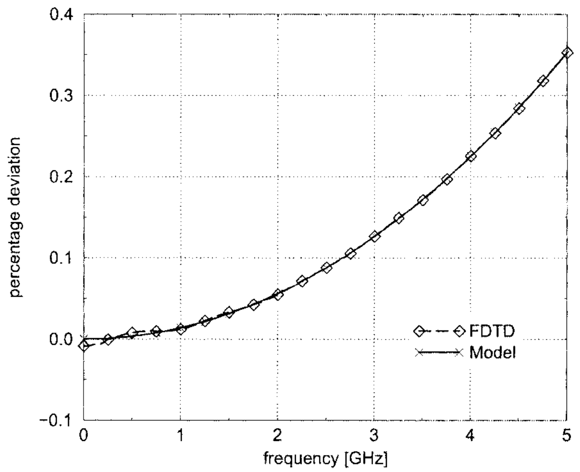

Fig. 3. Percentage deviation of the resistive layer reflection coefficient with respect to the ideal case (no discretization effects). Comparison of the FDTD with the model predictions.

The results are presented in the form of percentage deviation $\rho_{\%}$ of the reflection coefficient Γ_R with respect to the ideal case (1)

$$\rho_{\%} = 100\, \frac{|\Gamma_R| - |\Gamma_0|}{|\Gamma_0|}. \tag{11}$$

The percentage deviation defined above is illustrated in Fig. 3 for the two different cases where Γ_R is computed using either the model in (9) or the direct LE-FDTD simulation of the structure.

As it can be seen from the figure, the derived model agrees perfectly with the LE-FDTD computations, thus validating the expression of the cell resistance stated in (10).

IV. Conclusion

The numerical errors associated to the incorporation of lumped element into the FDTD framework have been rigorously evaluated in the simple but comprehensive case of a planar distribution of identical resistors. The results obtained show that the effective impedance of the resistors is purely real and depending on the space time discretization. These considerations could be used to predict the performances of absorbing boundary conditions based on the adoption of resistive layers.

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